問）$\int_0^{\frac{\pi}{3}} tan^7\theta\, d\theta$ を求めよ。

$\int_0^{\frac{\pi}{3}} (tan^{n+2}\theta + tan^n\theta)\, d\theta = \int_0^{\frac{\pi}{3}} tan^n\theta(1 + tan^2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{3}} tan^n\theta(tan\theta)'\, d\theta = [\frac{tan^{n+1}\theta}{n+1}]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}^{n+1}}{n+1}$

$I_n = \int_0^{\frac{\pi}{3}} tan^n\theta\, d\theta$ とおくと

$I_{n+2} + I_n = \frac{\sqrt{3}^{n+1}}{n+1}$ より　$I_{n+2} = -I_n + \frac{\sqrt{3}^{n+1}}{n+1}(n \geq 1)$

$n = 1, 3, 5$ と代入すると　$I_3 = -I_1 + \frac{\sqrt{3}^2}{2} \cdots (1)$

$I_5 = -I_3 + \frac{\sqrt{3}^4}{4} \cdots (2)$　$I_7 = -I_5 + \frac{\sqrt{3}^6}{6} \cdots (3)$

(1)(2)(3) より

$I_7 = -I_5 + \frac{27}{6} = -(-I_3 + \frac{9}{4}) + \frac{27}{6} = -I_1 + \frac{3}{2} - \frac{9}{4} + \frac{27}{6}$

$= -I_1 + \frac{15}{4}$

$I_1 = \int_0^{\frac{\pi}{3}} tan\theta\, d\theta = \int_0^{\frac{\pi}{3}} \frac{(-cos\theta)'}{cos\theta}\, d\theta = [-log|cos\theta|]_0^{\frac{\pi}{3}}$

$= log2$ なので

$\int_0^{\frac{\pi}{3}} tan^7\theta\, d\theta = \frac{15}{4} - log2$